The amplitude w(x, y) of the displacement of the medium outside the elastic inclusion is found from (1.4) by using (2.5) and (2.6).

Since the only assumptions made in deriving (2.5) and (2.6) were that ε is small and c, $b \gg 1$, these equations are valid over the whole region, including the boundary. In (2.5) only the case y = const and $x \to \infty$ must be excluded. In this case the expression for $w_1(x, y)$ has a somewhat different form. If it is required to find the distribution of the wave field inside the elastic inclusion, we obtain by proceeding as above

$$w_3(r,\varphi) = -\frac{ap_0}{2\theta_1\mu_1}\sqrt{\frac{\varepsilon}{2\pi\theta}}\left[\frac{2P_0I_0(\theta_1 r)}{I_0(\theta_1)(\varkappa G_0 - A_0)} - \frac{A_1I_1(\theta_1 r)(B_1\cos\varphi + B_2\sin\varphi)}{(\varkappa G_1 - A_1)\left(I_0(\theta_1) - \frac{1}{\theta_1}I_1(\theta_1)\right)}e^{i\frac{\pi}{4}}\right] + O(\varepsilon).$$

Thus, our proposed method permits the derivation of expressions for the wave field over practically the whole region under investigation which are rather simple to analyze. It should also be noted that this method can be employed without change to treat a similar problem in two or three dimensions. In doing this only the awkwardness of the calculations is substantially increased. Thus, in treating a similar problem in two dimensions it is necessary in the first stage to solve a system of six rather than three integral equations. However, all the basic properties of the elements of the system treated in the present article are retained. In calculating wave fields the integrals and sums which arise are of the same type as in the above treatment.

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DETERMINATION OF STRESSES IN AN INFINITE PLATE WITH BROKEN OR BRANCHING CRACK

P. N. Osiv and M. P. Savruk

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In numerical solution of the singular integral equations which arise in two-dimensional elasticity theory problems for bodies with internally smooth curvilinear sections, the mechanical quadrature method, based on Gauss–Chebyshev quadrature expressions, is employed. Considering a piecewise-smooth crack as a limiting case of a system of smooth sections [1-3], having common points, we arrive at a system of singular integral equations with generalized singular integrands, containing fixed singularities together with the Cauchy integrand. Such equations can also be solved by the mechanical quadrature method, although more complex quadrature expressions are required (for example, Gauss–Jacoby expressions), which consider the singularity of the solution at the nodes of the section contour. Below, using the example of a broken, branching crack in an infinite plate, we present a simplified technique for numerical solution of the integral equations for piecewise-smooth sections using Gauss–Chebyshev expressions. The solution singularity at the angular point or branching point is considered inexactly, so that such a solution is only effective when it is not necessary to determine the stressed state in the vicinity of such points. In particular, the proposed solution technique will be used to determine the stress intensity coefficients at the peaks of a piecewise-smooth crack.

1. Basic Assumptions. Within an infinite plane having a related Cartesian coordinate system xOy, let there be a system of N + 1 rectilinear sections L_n, located along segments $|x_n| \leqslant l_n$ of the local coordinate axes O_nx_n (n = 0, 1, ..., N). In the system xOy the origin

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O_n is located at the points $z = z_n^0$, and the axes $O_n x_n$ form angles α_n with the Ox axis. The edges of the cracks are loaded by a self-equalizing load $p_n(x_n)$, and stress is absent at infinity. Then the planar problem of elasticity theory for such a region reduces to a system of singular integral equations [4]:

$$\sum_{k=0}^{N}\int_{-l_k}^{l_k}\left[K_{nk}(t_k, x_n)\, g_k'(t_k) + L_{nk}(t_k, x_n)\,\overline{g_k'(t_k)}\right] dt_k = \pi p_n(x_n),\ |x_n| \leqslant l_n,\ n = 0, 1, \ldots, N, \tag{1.1}$$

where

$$K_{nk}(t_k, x_n) = \frac{e^{i\alpha_k}}{2}\left(\frac{1}{T_k - X_n} + \frac{e^{-2i\alpha_n}}{\overline{T}_k - \overline{X}_n}\right); \tag{1.2}$$

$$L_{nk}(t_k, x_n) = \frac{e^{-i\alpha_k}}{2}\left(\frac{1}{\overline{T}_k - \overline{X}_n} - \frac{T_k - X_n}{(\overline{T}_k - \overline{X}_n)^2}e^{-2i\alpha_n}\right);$$

$$T_k = t_k e^{i\alpha_k} + z_k^0;\ X_n = x_n e^{i\alpha_n} + z_n^0.$$

If the sections L_n ($n = 0, 1, \ldots, N$) are isolated, then the solution of system (1.1) must satisfy an additional $N + 1$ conditions, ensuring uniqueness of the displacements around each individual contour. In the case of a system of intersecting contours L_n ($n = 0, 1, \ldots, N$), forming an open contour $L_0 + L_1 + \ldots + L_N$, we obtain one condition [2], which ensures uniqueness of the displacements around the contour $L_0 + L_1 + \ldots + L_N$.

2. Two-Branched Broken Crack. We will consider a two-branch broken crack ($N = 1$) formed by two rectilinear sections L_0 and L_1. Along the segment $|x| \leqslant l$ ($l = l_0$) of the Ox axis we have the basic section L_0, from the right side of which, at an angle α to the Ox axis there exists a lateral section L_1, $2l_1$ in length (Fig. 1). The conditions for uniqueness of the displacements have the form

$$\int_{-l}^{l} g_0'(t_0)\, dt_0 + e^{i\alpha}\int_{-l_1}^{l_1} g_1'(t_1)\, dt_1 = 0. \tag{2.1}$$

Considering that in the given case $\alpha_0 = 0$, $z_0^0 = 0$, $\alpha_1 = \alpha$, $z_1^0 = l(1 + \varepsilon e^{i\alpha})$, $\varepsilon = l_1/l$, we write system (1.1) and condition (2.1) in normalized form:

$$\int_{-1}^{1}\left[\frac{\varphi(\xi)}{\xi - \eta} + M_{01}(\xi, \eta)\,\varphi_1(\xi) + N_{01}(\xi, \eta)\,\overline{\varphi_1(\xi)}\right] d\xi = \pi\sigma(\eta),\ |\eta| < 1, \tag{2.2}$$

$$\int_{-1}^{1}\left[M_{10}(\xi, \eta)\,\varphi(\xi) + N_{10}(\xi, \eta)\,\overline{\varphi(\xi)} + M_{11}(\xi, \eta)\,\varphi_1(\xi) + N_{11}(\xi, \eta)\,\overline{\varphi_1(\xi)}\right] d\xi = \pi\sigma_1(\eta),\ |\eta| < 1;$$

$$\int_{-1}^{1}\varphi(\xi)\, d\xi + \varepsilon e^{i\alpha}\int_{-1}^{1}\varphi_1(\xi)\, d\xi = 0. \tag{2.3}$$

In Eqs. (2.2), (2.3) and below the following notation is used:

$$M_{nk}(\xi, \eta) = l_k K_{01}(l_k\xi,\ l_n\eta);\ N_{nk}(\xi, \eta) = l_k L_{01}(l_k\xi,\ l_n\eta);$$

$$l_2 = l_3 = l_4 = l_1;\ z_2^0 = -z_1^0;\ z_3^0 = -\overline{z}_1^0;\ z_4^0 = \overline{z}_1^0;$$

$$\varphi(\xi) = g_0'(l\xi);\ \varphi_1(\xi) = g_1'(l_1\xi);\ \sigma(\eta) = p_0(l\eta);\ \sigma_1(\eta) = p_1(l_1\eta).$$

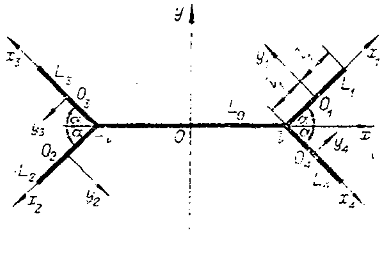

Fig. 1

It is evident from Eq. (1.2) that the integrands $M_{01}(\xi, \eta)$, $N_{01}(\xi, \eta)$, $M_{10}(\xi, \eta)$ and $N_{10}(\xi, \eta)$ have immobile singularities, i.e., they are generalized integrands. Consequently, the functions $\varphi(\eta)$ and $\varphi_1(\eta)$ at the points $\eta = 1$ and $\eta = -1$ have singularities.
We take

$$\varphi(\eta)=\frac{v(\eta)}{(1+\eta)^{1/2}(1-\eta)^{\beta}},\quad \varphi_1(\eta)=\frac{v_1(\eta)}{(1+\eta)^{\beta}(1-\eta)^{1/2}}$$

and assume that $v(\pm 1)$ and $v_1(\pm 1)$ are not equal to zero.

On the basis of the Sokhotskii–Plemel expressions for Cauchy-type integrals

$$\Phi(z)=\frac{1}{2\pi}\int\limits_L \frac{g'(t)}{t-z}\,dt,\ L=L_0+L_1$$

we have

$$ig'(t)=\Phi^{+}(t)-\Phi^{-}(t),\ t\in L, \tag{2.4}$$

which is also valid at an angular point [5]. It follows from Eq. (2.4) that the singularity of the functions $\varphi(\eta)$ and $\varphi_1(\eta)$ at the points $\eta = 1$ and $\eta = -1$ is the same as the maximum singularity of the complex potential $\Phi(z)$ at the angular points of the wedgelike regions into which the crack divides the body, i.e., for definition of the exponent β we have the characteristic equation [6]:

$$\sin[(1-\beta)(\pi+\alpha)]=-(1-\beta)\sin(\pi+\alpha),\quad 0<\beta<1.$$

Analysis of the roots of this equation shows (see, for example, [7]) that $\beta \le 1/2$. Consequently, the functions $\varphi(\eta)$ and $\varphi_1(\eta)$ can be written in the form $\varphi(\eta) = u(\eta)/\sqrt{1-\eta^2}$, $\varphi_1(\eta) = u_1(\eta)/\sqrt{1-\eta^2}$, assuming that $u(1) = 0$, $u_1(-1) = 0$.

Now applying to Eq. (2.2) and condition (2.3) the Gauss–Chebyshev quadrature expressions [8]

$$\int\limits_{-1}^{1}\frac{u(\xi)\,d\xi}{\sqrt{1-\xi^2}(\xi-\eta_m)}=\frac{\pi}{N_1}\sum_{k=1}^{N_1}\frac{u(\xi_k)}{\xi_k-\eta_m}\quad \left(\eta_m=\cos\frac{m\pi}{N_1},\ m=1,2,\ldots,N_1-1\right),$$

$$\int\limits_{-1}^{1}\frac{u(\xi)}{\sqrt{1-\xi^2}}\,d\xi=\frac{\pi}{N_1}\sum_{k=1}^{N_1}u(\xi_k),\ \xi_k=\cos\frac{2k-1}{2N_1}\pi,$$

we arrive at a system of $2N_1 - 1$ algebraic equations for determination of $2N_1$ unknowns $u(\xi_k)$ and $u_1(\xi_k)$. To obtain a closed system, we add one of the equations

$$u(1)=-\frac{1}{N_1}\sum_{k=1}^{N_1}(-1)^k u(\xi_k)\operatorname{ctg}\frac{2k-1}{4N_1}\pi=0, \tag{2.5}$$

$$u_1(-1)=\frac{1}{N_1}\sum_{k=1}^{N_1}(-1)^{N_1+k}u_1(\xi_k)\operatorname{tg}\frac{2k-1}{4N_1}\pi=0.$$

Calculations show that the efficiency of the numerical solution is practically unaffected by which expression of Eq. (2.5) is chosen for this purpose.

The stress intensity coefficients at the left $K_{1,2}^{-}$ and right $K_{1,2}^{+}$ sides of the broken crack have the form [4]

$$K_1^{+}-iK_2^{+}=\frac{\sqrt{l_1}}{N_1}\sum_{k=1}^{N_1}(-1)^k u_1(\xi_k)\operatorname{ctg}\frac{2k-1}{4N_1}\pi,$$

$$K_1^{-}-iK_2^{-}=\frac{\sqrt{l_1}}{N_1}\sum_{k=1}^{N_1}(-1)^{N_1+k}u(\xi_k)\operatorname{tg}\frac{2k-1}{4N_1}\pi.$$

We will obtain a solution for the case where the edges of the crack are free of load, while at infinity the plane is in tension from external stresses p and q, acting in mutually perpendicular directions, with p directed at an angle γ to the Ox axis. By superposition this problem reduces to system (2.2) with right-hand side

$$\sigma(\eta)=-(1/2)[p+q-(p-q)e^{2i\gamma}],$$
$$\sigma_1(\eta)=-(1/2)[p+q-(p-q)e^{2i(\gamma-\alpha)}].$$

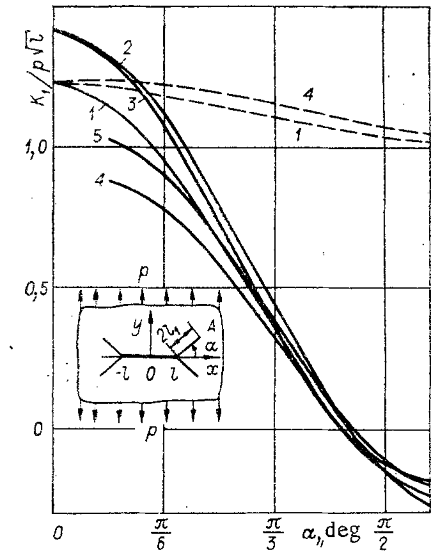

Fig. 2

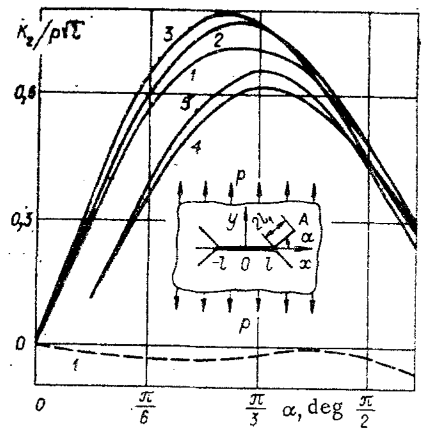

Fig. 3

For ε = 0.5, Figs. 2-5 show the stress intensity coefficients K_1 and K_2 in units of $p\sqrt{l}$ as functions of the angle at which the lateral crack is oriented for the case of uniaxial ($q = 0$, $\gamma = \pi/2$, Figs. 2 and 3) and omnidirectional ($p = q$, Figs. 4 and 5) tension. The solid lines describe the right-hand side of the crack (side A), and the dashed lines, the left. Curves 1 correspond to the case of a two-branched broken crack.

3. Three-Branch Broken Crack. We assume that the broken crack consists of three rectilinear segments L_0, L_1, and L_2 (Fig. 1). Since in this case the stressed state of the body is symmetric relative to the center of the crack ($z = 0$), we obtain $g_2'(t_2) = g_1'(t_1)$. The system of three ($N = 2$) integral equations (1.1) with consideration of the symmetry involved is written in the form of Eq. (2.2), where to the integrands $M_{01}(\xi, \eta)$, $N_{01}(\xi, \eta)$, $M_{11}(\xi, \eta)$ and $N_{11}(\xi, \eta)$ we must add the functions $M_{02}(\xi, \eta)$, $N_{02}(\xi, \eta)$, $M_{12}(\xi, \eta)$ and $N_{12}(\xi, \eta)$, respectively. The condition for uniqueness of the displacements then takes on the form

$$\int_{-1}^{1} \varphi(\xi)\, d\xi = 0.$$

Numerical results for this problem are illustrated by Figs. 2-5 (curves 2).

We will now consider a broken crack consisting of segments L_0, L_1, and L_3, where the lateral sections are symmetric about the axis Oy (Fig. 1). Then from the symmetry of the problem it follows that $g_3'(x_3) = g_1'(x_1)$. In this case we also arrive at system (2.2), in

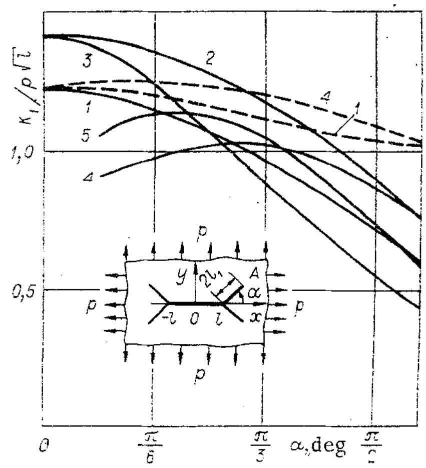

Fig. 4

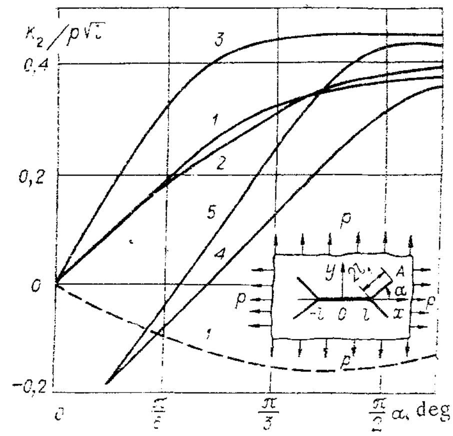

Fig. 5

which we add to the integrands $M_{01}(\xi, \eta)$, $N_{01}(\xi, \eta)$, $M_{11}(\xi, \eta)$ and $N_{11}(\xi, \eta)$ we add, respectively, $N_{03}(\xi, \eta)$, $M_{03}(\xi, \eta)$, $N_{13}(\xi, \eta)$ and $M_{13}(\xi, \eta)$. The uniqueness condition is written in the form

$$\int_{-1}^{1} \varphi(\xi)\, d\xi + 2\varepsilon i \,\mathrm{Im}\left[e^{i\alpha} \int_{-1}^{1} \varphi_1(\xi)\, d\xi\right] = 0.$$

The stress intensity coefficients K_1 and K_2 are shown as functions of angle α for the three-branched crack in Figs. 2-5 (curves 3).

4. Branching Crack. Let an infinite plate be weakened by a main section L_0, from the right side of which two lateral sections L_1 and L_4 exit (Fig. 1). The integral equations of the problem for such a region have the form of Eq. (1.1) with N = 4 and $g_2'(x_2) = g_3'(x_3) = 0$. Numerical solution of integral equations (1.1) can be performed by the same technique as in the previous case of broken cracks.

Considering the condition $g_4'(x_4) = g_1'(x_1)$, we arrive at system (2.2), the solution of which must satisfy the equation

$$\int_{-1}^{1} \varphi(\xi)\, d\xi + 2\varepsilon \,\mathrm{Re}\left[e^{i\alpha} \int_{-1}^{1} \varphi_1(\xi)\, d\xi\right] = 0.$$

Here we add to the integrands $M_{01}(\xi, \eta)$, $N_{01}(\xi, \eta)$, $M_{11}(\xi, \eta)$ and $N_{11}(\xi, \eta)$, respectively, $N_{04}(\xi, \eta)$, $M_{04}(\xi, \eta)$, $N_{14}(\xi, \eta)$ and $M_{14}(\xi, \eta)$. Curves 4 of Figs. 2-5 show this case.

We note that the problem of uniaxial tension at infinity in a plane with branching crack was considered in a similar manner in [3]. The system of singular integral equations (1.1) was also used, and numerical solution was effected with the aid of Gauss and Lobatto quadrature expressions. The results obtained herein agree well with the data of [3].

We will consider a more general case of the branching crack, where from both ends of the main crack L_0 there exit two lateral cracks: L_1, L_4 and L_2, L_3 (see Fig. 1). It follows from symmetry that $g_2'(x_2) = g_3'(x_3) = g_4'(x_4) = g_1'(x_1)$. Then system (1.1) (N = 4) leads to Eq. (2.2), where we add to the integrands $M_{01}(\xi, \eta)$, $N_{01}(\xi, \eta)$, $M_{11}(\xi, \eta)$ and $N_{11}(\xi, \eta)$, the quantities $M_{02}(\xi, \eta) + N_{03}(\xi, \eta) + N_{04}(\xi, \eta)$, $N_{02}(\xi, \eta) + M_{03}(\xi, \eta) + M_{04}(\xi, \eta)$, $M_{12}(\xi, \eta) + N_{13}(\xi, \eta) + N_{14}(\xi, \eta)$ and $N_{12}(\xi, \eta) + M_{13}(\xi, \eta) + M_{14}(\xi, \eta)$. The uniqueness condition has the form

$$\int_{-1}^{1} \varphi(\xi)\, d\xi = 0.$$

The stress intensity coefficients obtained for this last case (curves 5 of Figs. 2-5) with uniaxial tension of the plate practically coincide with the results of [10].

5. Crack with Infinitely Small Branches. In the mechanics of failure, in particular, in constructing energy criteria for crack propagation, solutions of the above problems for the limiting case in which the ratio of the branch lengths to that of the main crack is infinitely small are of principal significance. In this case the stress intensity coefficients at the edges of the small branches can be represented in the form $K_i = k_1 K_{i1}(\alpha) + k_2 K_{i2}(\alpha)$ ($i = 1, 2$), where k_1 and k_2 are the intensity coefficients for the main crack in the absence of branches.

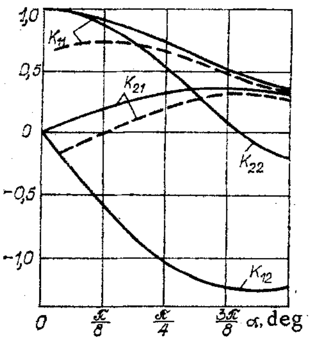

Fig. 6

The convergence of the numerical solution of system (2.2) to an exact value degrades as the ratio of side crack length to main crack length decreases. Therefore in the limiting case as $l_1/l \to 0$ a numerical solution cannot be obtained directly. For a two-branch broken crack (Fig. 6, solid lines) and a three-branched crack (Fig. 6, dashed lines) the values of $K_{ij}(\alpha)$ were calculated by extrapolation from numerical data obtained for l_1/l = 0.01, 0.02.

Similar functions $K_{ij}(\alpha)$ were presented in [11]. In the case of the broken crack there is quite good agreement between the results obtained and the data of [11] (maximum relative deviation does not exceed 6%), with significantly greater differences for the branching crack. For this last case, [12] presents the dependence of stress intensity coefficients on angle α for l_1/l = 0.1. We note that for such an l_1/l value the intensity coefficients calculated by solution of Eq. (2.2) practically coincide with the data of [12].

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OPENING OF A NATURAL MACROCRACK

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The shortcomings of the simplest models of macrocracks have been noted many times in the literature. Attempts to construct complete models reduce to selecting some hypothesis concerning the behavior of the medium at the tips of the crack [1, 2], but the process of formation of real macrocracks was not given the proper attention.

A model of natural macrocracks, which takes into account the presence of residual compressive stresses arising at the tip of a crack as it is formed and opposing the opening up of the macrocrack, was proposed in [3, 4]. The purpose of this investigation is to provide experimental justification of the model proposed.

1.To investigate the mechanisms involved in opening up of a natural macrocracks, we prepared a rectangular specimen consisting of SO-95 Plexiglas, to which we gave a matted

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